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Some Problems in Computational Representation Theory

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Dedicated to Professor G. E. Wall on the occasion of his 65th birthday

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Modern computers and computer algebra systems yield powerful tools for mathematical research. In this article, some applications of computer algebra packages and scientific software in the theory of modular representations of finite groups are described. Ten mathematical problems are mentioned, the solutions of which require new algorithms and most likely extensions of the present mathematical software. It is hoped that the corresponding computer experiments stimulate further mathematical research. Furthermore, they will show the importance of the development of computer algebra for theoretical mathematics.

1. Introduction

Construction of the irreducible representations of a finite group G requires a detailed knowledge of the group structure of G. For the study of concrete examples it is often extremely useful to apply the algorithms contained in the general computer algebra systems. So far, the systems CAYLEY of Cannon (1984) and CAS of Neubüser, Pahlings & Plesken (1984) have played an important role in the study and application of special group representations.

In addition, the computer may also be used for the computation of the generic character tables of some finite groups of Lie type $G_n(q)$ defined over finite fields $GF(q)$ with q elements, as long as the Lie rank n of $G_n(q)$ is not too large. Since these computations require the symbolic manipulation of algebraic polynomials in one indeterminate q over a ring of algebraic integers, general computer algebra systems such as MAPLE (Char *et al.*, 1985) and SCRATCHPAD II (Jenks, 1984) have recently found applications to the representation theory of finite groups of Lie type. Cohen's system LIE (Cohen, 1989) is also very useful.

In the following section, some of the noteworthy achievements are described. Furthermore, several other computational problems are mentioned, e.g. the computation of the value $R^G_{T,\theta}(g)$ of a Deligne–Lusztig character at an element g of $G_n(q)$. Although there is a deterministic algorithm for the evaluation of $R^G_{T,\theta}$ at g, in general, the concrete computation presents tremendous difficulties.

Several sporadic simple groups have been constructed in terms of irreducible modular representations over small fields (Gorenstein, 1982). One of the most ingenious constructions is that used to construct the simple Janko group J_4 by Benson *et al.* as a specific subgroup of the general linear group $GL(112, 2)$ (Gorenstein, 1982). Variations of

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this method developed by Parker & Wilson (1990) and Gollan (1990) are described in section 3. The central problem here may be stated simply: How does one find explicit matrix generators for a given finite group G? Often they may be used to determine a set of defining relations satisfied by these generators.

The correspondences of Brauer, Green and Fong yield very powerful tools for the general modular representation theory of arbitrary finite groups G over fields F with characteristic $p > 0$ dividing the order $|G|$. However, many open problems remain in the subject. For example, no criteria are known for subgroups or indecomposable modules to be vertices or sources of simple FG-modules, respectively. In his book, Feit (1982, p. 121) writes: "Virtually nothing is known in this connection". It is therefore important to study numerous examples in great detail. For this, the new algorithms of Schneider (1990) for the computation of the vertices and sources of indecomposable modules M of dimension $\dim_F M \leq 300$ are very helpful. They will be described in section 4. Effective use of these methods was made by Gollan (1990), who recently determined the sources of the simple 5-modular representations of the simple Tits-group ${}^2F_4(2)'$. His results are summarised in section 5, where, additionally, some related open problems on modular representations of groups with a T.I. Sylow p-subgroup are mentioned.

Most of the modular irreducible representations of the sporadic simple groups are now known. Parker's MEAT-AXE program and the MOC-computer package of Hiss *et al.* (in preparation) provide powerful tools for tackling this problem. Nevertheless, several hard cases still remain unsolved. In order to describe the irreducible modular and indecomposable projective representations of a p-block B with cyclic defect group, it is necessary to determine the Brauer tree of B. Although the book by Hiss & Lux (1989) contains most of the trees of the cyclic p-blocks of the simple sporadic groups, there are still some cases left open. In view of the strength of the theoretical results by Brauer and Dade, this is somewhat astonishing.

The source of the basic difficulties in all these computational problems, arising from modular representation theory, lies in the cost of performing computations with $n \times n$ matrices over finite fields $GF(q)$. Here, the sizes of n and q impose serious restrictions for successful computer calculations. This is even more so for the symbolic computations which are necessary for the problems mentioned in section 2 concerning the determination of the character tables of some finite groups of Lie type.

It is therefore hoped that computer experiments will provide enough evidence for possible general mathematical results which then can be proved completely by means of theoretical methods.

Concerning terminology and notation we refer to the books by Benson (1984), Carter (1985), Feit (1982), Isaacs (1976) and the author's lecture notes (Michler, 1989).

2. Character Tables of Finite Groups of Lie Type

In 1907 Schur determined the complete character tables of the finite two-dimensional projective linear groups $PGL(2, q)$ and the two-dimensional special linear groups $SL(2, q)$ defined over finite fields $GF(q)$ with q elements. Almost 40 years later, Steinberg solved the same problem for the three- and four-dimensional linear groups. In 1973 Simpson and Frame treated the case of all three-dimensional unitary groups $PSU(3, q^2)$ and $SU(3, q^2)$. One year later, Chang and Ree determined the complete character table of the groups $G_2(q)$. Since then additional character tables have been calculated.

In 1976 Deligne and Lusztig published their famous paper "Representations of reductive groups over finite fields" in the *Annals of Mathematics*. In this paper they present general methods for the construction of families of irreducible characters of an arbitrary finite group of Lie type. Further advances have subsequently been made by Lusztig, and are presented in his book (Lusztig, 1984). In particular, the degrees of the irreducible representations of the finite groups of Lie type are now completely known. Furthermore, many character values can be computed. Since such a group has huge numbers of conjugacy classes in general, it is clear that its complete character table can only be determined if its Lie rank is small. Such a case has been dealt with by Deriziotis and the author (Deriziotis & Michler, 1987), where the character table of the Steinberg triality $^3D_4(q)$ is derived.

At present the character tables of the exceptional groups of Lie type $F_4(q)$, $E_n(q)$ for $n = 6$, 7 and 8 have not been completed for arbitrary q. In the Atlas (Conway *et al.*, 1985) the character tables for $F_4(2)$ and $^2E_6(2)$ are given. Recently, Fischer (Bielefeld) has calculated the character table of $E_6(2)$. However, the tables for $E_7(2)$ and $E_8(2)$ are not yet known. For arbitrary q, the number of conjugacy classes for finite group of Lie type $E_n(q)$ with rank n is a polynomial in q with rational coefficients $a_i \in \mathbb{Z}$ and degree n. The difficulty of determining the complete generic character table of $E_7(q)$ and $E_8(q)$ is indicated by:

PROBLEM 1. Compute the rational coefficients a_i of the polynomials

$$p_n(q) = q^n + \sum_{i=0}^{n-1} a_i q^i$$

describing the numbers of conjugacy classes of the simple groups $E_n(q)$ for $n = 7, 8$.

An understanding of the computational difficulties of this and some related partial problems requires the following notations and general results about finite groups of Lie type.

Every finite Chevalley or twisted group of Lie type considered here is the group G_σ of fixed points of an endomorphism σ of a simple connected algebraic group G defined over a finite field $F = GF(q)$ with $q = p^m$ elements, where the prime number $p > 0$ is the characteristic of F. The simply connected covering group of G is denoted by G_{sc} and the adjoint group by G_{ad}, the dual of G_{sc}. For further explanation, see Carter (1985).

Let T be a σ-stable maximal torus of G. The maximal torus of G_σ is a subgroup of the form $T_\sigma = G_\sigma \cap T$. The quotient $W_\sigma = (N_G(T)/T)_\sigma$ is the Weyl group of the maximal torus T_σ of G_σ. It acts on T_σ and its character group $\hat{T}_\sigma$. The set $C_{W_\sigma}(s) = \{\omega \in W_\sigma \mid s^\omega = s\}$ is called the *isotropy group* of the semi-simple element s of T_σ. If $C_{W_\sigma}(s) = 1$, then $s \in T$ is said to be in *general position*. An element s of a maximal torus T is called *regular* if T coincides with the connected component $C_G(s)^0$ of $C_G(s)$, where H^0 denotes the connected component of the group H. Elements $s \in G_\sigma$ which are in general position, are regular; the converse is true in simply connected groups, but not in general.

Let W be the Weyl group of a fixed σ-stable maximally split torus T_0 of G. Then $W = N_G(T_0)/T_0$ is uniquely determined by G up to G-conjugation. By Carter (1985, p. 84) the G_σ-conjugacy classes of σ-stable maximal tori T of G are in bijective correspondence with the σ-conjugacy classes of W. Two elements $\omega_1, \omega_2 \in W$ are σ-conjugate, if $\omega_2 = \omega\omega_1\sigma(\omega)^{-1}$, for some $\omega \in W$.

A maximal torus T corresponding to the identity class of the Weyl group W is called a *maximal split torus* of G_σ. It is denoted by T_0.

Let G be a simply connected reductive algebraic group and σ an endomorphism of G

such that its group G_σ of fixed points is finite. If G has semi-simple Lie rank n, then G_σ has $|Z_\sigma^0|q^n$ semi-simple conjugacy classes by Theorem 3.7.6. of Carter (1985). Here Z^0 denotes the connected centre of G. But in order to compute the total number of conjugacy classes of G_σ, it is necessary to find the precise answer to the following.

PROBLEM 2. Let T be a maximal torus of G_σ. Determine the number of G_σ-conjugacy classes of regular elements having a representative in T.

Recently, Janiszczak computed these numbers for $E_6(q)$ and $E_7(q)$ using a machine. Similar computations for $E_8(q)$ are in progress.

The Deligne–Lusztig generalised character $R_{T,\theta}$ of a finite group G_σ of Lie type are defined in Chapter 7 of Carter (1985). Here, T denotes a σ-stable maximal torus of the connected reductive group G, and θ is a linear character of the finite abelian subgroup T_σ of G_σ. The importance of the generalised characters $R_{T,\theta}$ is best explained by Corollary 7.5.8 of Carter (1985): for every irreducible character χ of G_σ, there exists a generalised character $R_{T,\theta}$ such that the inner product $(R_{T,\theta}, \chi) \neq 0$. Furthermore, Corollary 7.3.5. of Carter (1985) asserts that $\pm R_{T,\theta}$ is an irreducible character of G_σ if $\theta \in \mathrm{Hom}(T_\sigma, \mathbb{C}) = \hat{T}_\sigma$ is in general position. Since such linear characters $\theta \in \hat{T}_\sigma$ correspond to regular elements s of T_σ under the duality between T_σ and $\hat{T}_\sigma$, this consequence of the Deligne–Lusztig theory again demonstrates the importance of Problem 2 for the construction of irreducible characters of finite groups G_σ of Lie type.

It is well known that each element $g \in G_\sigma$ has a unique Jordan decomposition $g = su = us$, where u is a unipotent and s is a semi-simple element of G_σ. The Green function Q_T is defined by $Q_T(u) = R_{T,1}(u)$ for every unipotent element $u \in G_\sigma$. According to Carter (1985), all its values $Q_T(u)$ are rational integers.

If s is a semi-simple element of the reductive group G, then $C^0(s)$ denotes the connected centraliser of s in G. A very important character formula in the Deligne–Lusztig theory is the following:

Let $g \in G_\sigma$ have Jordan decomposition $g = su = us$, where s is semi-simple and u is unipotent. Then by (Carter 1985),

$$R_{T,\theta}(g) = \frac{1}{|C^0(s)_\sigma|} \sum_{\substack{x \in G_\sigma \\ x^{-1}sx \in T_\sigma}} \theta(x^{-1}sx) Q^{C^0(s)}_{xTx^{-1}}(u) \qquad (*)$$

Although formula (*) yields a complete algorithm for the evaluation of the Deligne–Lusztig character $R_{T,\theta}$ at an element $g \in G_\sigma$, it is rather complicated to compute $R_{T,\theta}(g)$, even in the case of small Lie rank n. It is first necessary to solve the following.

PROBLEM 3. Find the values $Q_T(u)$ of the Green functions Q_T at unipotent elements $u \in G_\sigma$.

In the books of Carter (1985) and Lusztig (1984), several methods for the solution of that problem have been given. Shoji (1982) has computed the Green polynomials for $F_4(q)$. Beynon & Spaltenstein (1984) solved the same problem for $E_n(q)$, $n = 6$, 7 and 8. As can be seen from their paper, the computer has played an essential role. Using the computer algebra systems CAYLEY and REDUCE, Lambe & Srinivasan (1978) have computed the Green functions of some finite classical groups of type C_n $(2 \leq n \leq 5)$, B_n $(3 \leq n \leq 5)$ and D_n $(n = 4, 5)$ for q odd.

Having solved Problem 3 for a given group G_σ it is then still hard to find efficient algorithms for the solution of:

PROBLEM 4. Find the values $R_{T,\theta}(g)$, $g \in G_\sigma$, using formula (*).

The case of semi-simple elements $g = s$ does not involve the evaluation of the Green functions because of Proposition 7.5.3. of Carter (1985). Nevertheless, in order to be able to decide whether $x^{-1}sx \in T_\sigma$, it is necessary to have an algorithm for finding a complete system of double coset representatives x_i of T_σ. Also, the evaluation of the linear characters θ requires extension of the present tools for computing with algebraic numbers. Thus far, these computational problems have only been solved for groups $G_n(q)$ with small Lie rank $n \leq 5$.

In 1963 Ennola determined the complete character table of the unitary group $U(3, q^2)$. Ten years later, Simpson & Frame (1973) computed the character table of the special unitary group $SU(3, q^2)$. Using the computer algebra system MAPLE, Geck (1987) was able to find the restrictions of the irreducible characters of $U(3, q^2)$ to $SU(3, q^2)$ automatically. In fact, his method is a very efficient way to compute the character table of $SU(3, q^2)$. Since the character table of $U(4, q^2)$ is known, MAPLE can also be used to compute the character table of $SU(4, q^2)$.

3. Constructions of Representations

In this section some methods for constructing group representations are reviewed. Throughout this section (F, R, S) will denote a splitting p-modular system for the finite group G (Feit, 1982 or Michler, 1989).

DEFINITION 1. Let $A \in \{\mathrm{F}, R, S\}$ and let U be a subgroup of G. If 1_U denotes the trivial AU-module, then $M = (1_U)^G = 1_U \otimes_{AU} AG$ is called the *permutation module of* G with respect to U.

LEMMA 1.
(*a*) $\dim_A(1_U)^G = |G:U|$.
(*b*) *If* $\kappa: G \to GL(\dim_A M, A)$ *denotes the group representation of* G *afforded by* $M = (1_U)^G$, *then each* $\kappa(g)$, $g \in G$, *is a permutation matrix.*
(*c*) *The endomorphism rings of the permutation modules* $M_\mathrm{F} = 1_U \otimes_{\mathrm{F}U} \mathrm{F}G$, $M_R = 1_U \otimes_{RU} RG$ *and* $M_S = 1_U \otimes_{SU} SG$ *satisfy the conditions:*

$$\mathrm{End}_{\mathrm{F}G}(M_\mathrm{F}) \cong \mathrm{F} \otimes_R \mathrm{End}_{RG}(M_R),$$
$$\mathrm{End}_{SG}(M_S) \cong S \otimes_R End_{RG}(M_R).$$

(*d*) $M_\mathrm{F} \cong \mathrm{F} \otimes_R M_R$ *and* $M_S \cong S \otimes_R M_R$.

In general, it may be difficult to find a permutation representation $(1_U)^G$ of a group G by means of the Todd–Coxeter algorithm, see Cannon (1984). In order to determine the ring theoretical structure of the endomorphism ring $\mathrm{End}_{\mathrm{F}G}[(1_U)^G]$ of a permutation module $(1_U)^G$ of a subgroup U of G, it suffices to compute a complete system of representatives for the double cosets of U. For this easier task it is necessary to solve:

PROBLEM 5. Find a deterministic algorithm for the computation of a complete system of representatives x for the U–U double cosets UxU of a subgroup U in the finite group G.

Apart from permutation modules, tensor product constructions are used. Theoretically, they are important because of the following classical result:

THEOREM 1. (*R. Brauer*). *Let $A \in \{F, S\}$ and let M be a faithful simple AG-module. Then every simple AG-module T is a composition factor of some tensor power $M^{\otimes n}$ of M.*

In practice, tensor products are very difficult to split as they correspond to modules having fairly large dimension. Even wedge powers $\hat{\imath}M$ of FG-modules M are, in general, too large for concrete computer computations.

When Benson, Conway, Parker, Norton and Thackray constructed the simple Janko group J_4 as a subgroup of $GL(112, 2)$, they introduced a method for extending representations from pairs of proper subgroups of a group G to the whole group G. This technique has been slightly altered and generalised by Gollan (1989). In order to state his version of the result, the following notation is needed.

DEFINITION 2. Let $\kappa: G \to GL(n, \mathrm{F})$ be an n-dimensional representation of the finite group G over the field F. Let U be a subgroup of G and $\kappa_{|U}$ the restriction of κ to U. Then $\kappa(U)$ is a subgroup of $GL(n, \mathrm{F})$. The *centraliser of* $\kappa_{|U}$ *in* $GL(n, \mathrm{F})$ is defined by $C_{GL(n,\mathrm{F})}(\kappa_{|U}) = C_{GL(n,\mathrm{F})}(\kappa(U))$. In particular, $C_{GL(n,\mathrm{F})}(\kappa) = C_{GL(n,\mathrm{F})}(\kappa(G))$.

THEOREM 2. (*Gollan*). *Let $U_1 = \langle x_1, x_2, \ldots, x_r \rangle$ and $U_2 = \langle y_1, y_2, \ldots, y_m \rangle$ be two subgroups of the finite group G such that G is generated by the x_i and y_j, $1 \le i \le r$, $1 \le j \le m$. Let F be a splitting field for G with characteristic $p > 0$ not dividing the order of $D = U_1 \cap U_2$. Suppose that for $s = 1, 2$, $\kappa_s: U_s \to GL(n, \mathrm{F})$ is a representation of U_s such that the restrictions $\kappa_{s|D}$ yield the same representation of D, i.e. $\kappa_{1|D} = \kappa_{2|D}$. Let*

$$k = |C_{GL(n,\mathrm{F})}(\kappa_{1|D}) : C_{GL(n,\mathrm{F})}(\kappa_1)|,$$

and let $T_1, T_2, \ldots, T_k \in GL(n, \mathrm{F})$ be a transversal for $C_{GL(n,\mathrm{F})}(\kappa_1)$ in $C_{GL(n,\mathrm{F})}(\kappa_{1|D})$. If κ_1 and κ_2 have a common extension $\kappa: G \to GL(n, \mathrm{F})$ to the whole group G, then there is a matrix $T \in \{T_u | 1 \le u \le k\}$ such that

$$\kappa(G) = \langle \kappa_1(x_1), \kappa_1(x_2), \ldots, \kappa_1(x_r), T\kappa_2(y_1)T^{-1}, \ldots, T\kappa_2(y_m)T^{-1} \rangle.$$

Furthermore, conjugation by T induces an isomorphism between $\kappa_{1|D}$ and $\kappa_{2|D}$.

REMARK. The above theorem can also be used to show that two given subrepresentations $\kappa_s: U_s \to GL(n, \mathrm{F})$, $s = 1, 2$, do not have a common extension κ to the whole group G. In this case there does not exist a transformation matrix T such that

$$\kappa(G) = \langle \kappa_1(x_1), \kappa_1(x_2), \ldots, \kappa_1(x_r), T\kappa_2(y_1)T^{-1}, \ldots, T\kappa_2(y_m)T^{-1} \rangle.$$

Of course, the resulting computation for large groups may be very expensive.

Conversely, if it is known that κ_1 and κ_2 extend to an irreducible representation κ of the whole group G, then it follows that there exist transformation matrices T such that the group $\kappa(G)$ is generated by the matrices $\kappa_1(x_i)$, $1 \le i \le r$, and the matrices $T\kappa_2(y_j)T^{-1}$, $1 \le j \le m$.

REMARK 2. Gollan (1990) has applied this theorem to construct the 26-dimensional 5-modular irreducible representation of the simple Tits group $G = {}^2\mathrm{F}_4(2)'$ having order $|G| = 2^{11}\, 3^3\, 5^2\, 13$. Here, G contains two conjugacy classes of maximal subgroups U_1 and

U_2 which are both isomorphic to the automorphism group of $PSL(3,3)$, i.e. $U_i \cong PSL(3,3){:}2$, $i = 1,2$. Their intersection $D = U_1 \cap U_2 \cong 3^{1+2}{:}D_8$ in the notation of the ATLAS, where 3^{1+2} denotes the extra-special 3-group of order 27 and exponent 3.

In this example, the index $k = 24^4$. For the details of the construction, see Gollan (1990).

REMARK 3. In the original construction of J_4 by Norton and Parker, the hypothesis $p \nmid |U_1 \cap U_2|$ is not assumed. Nevertheless, they were able to describe the indecomposable summands of the restrictions of the representations κ_1 and κ_2 to $U_1 \cap U_2$. In general, the structure of these summands of the restrictions will not be known. For this reason Gollan considered semi-simple restrictions.

However, there are several other successful applications of the Cambridge method without the restrictive hypothesis. In particular, the Lyons group Ly has recently been constructed as a subgroup of $GL(111, 5)$ by Parker & Wilson (1990). Other applications are described in this paper.

4. Induction and Restriction of Modular Representations

Throughout this section G will denote a finite group and F a finite field having characteristic $p > 0$ dividing the order $|G|$ of G. By the Krull–Schmidt theorem every finitely generated FG-module M can be uniquely decomposed into indecomposable non-isomorphic FG-submodules M_i with multiplicities m_i such that

$$M \cong \sum_{i=1}^{s} \otimes\, m_i M_i,$$

where $m_i M_i$ denotes the direct sum of m_i copies of M_i.

PROBLEM 6. Find an efficient algorithm for the decomposition of a finitely generated FG-module M into indecomposable direct summands M_i.

Of course, it suffices to find a complete set of orthogonal primitive idempotents e in the endomorphism ring $E = \mathrm{End}_{\mathrm{F}G}(M)$ with sum equal to the identity element $1 \in E$. The basic step is:

PROBLEM 7. Find a non-zero idempotent $e \neq 0$ in the endomorphism ring $E = \mathrm{End}_{\mathrm{F}G}(M)$ of an FG-module M.

It turns out that M is indecomposable if and only if the identity endomorphism of M is the only non-zero idempotent in $E = \mathrm{End}_{\mathrm{F}G}(M)$.

Schneider (1990) has recently given probabilistic algorithms for Problems 6 and 7. They work very well as long as $\dim_{\mathrm{F}} M \leq 300$. They have been installed in the computer algebra system CAYLEY.

The socle series of an FG-module M is a series

$$0 = \mathrm{soc}_0(M) < \mathrm{soc}_1(M) < \mathrm{soc}_2(M) < \ldots < \mathrm{soc}_s(M) = M$$

of submodules of M such that $\mathrm{soc}_i(M)/\mathrm{soc}_{i-1}(M)$ is the largest semi-simple submodule of $M/\mathrm{soc}_{i-1}(M)$ for $1 \leq i \leq s$. The integer s is called the *socle length* of M.

In practice, Parker's Meataxe may sometimes be used to compute the socle series of an FG-module M. Since the submodules $\mathrm{soc}_i(M)$ are invariant under all endomorphisms of

M, the socle series of M may provide an efficient way to solve Problems 6 or 7 in specific cases.

In modular representation theory, Green's theory of vertices and sources plays an important role. A *vertex* of an indecomposable FG-module M is a p-subgroup S of smallest order such that the FG-module $M{\downarrow_S}{\uparrow^G}$ has a direct summand isomorphic to M. In fact, $S = vx(M)$ is uniquely determined by M up to G-conjugacy. An indecomposable FS-module Q is a *source* of M if M is isomorphic to a direct summand of the induced FG-module $Q{\uparrow^G} = Q \otimes_{\mathrm{F}S} \mathrm{F}G$.

PROBLEM 8. Let M be an indecomposable FG-module. Find deterministic algorithms for the determination of the vertices $vx(M)$ and sources Q of M.

Schneider (1990) has developed algorithms for Problem 8 which are effective for modules M such that $\dim_{\mathrm{F}} M \le 300$. The algorithms developed by Schneider include an algorithm for the computation of indempotents in the endomorphism ring $E = \mathrm{End}_{\mathrm{F}G}(M)$ and an algorithm WRDSOL that returns any element of a group G as word in the given set $\{g_i \mid 1 \le i \le k\}$ of generators of G. This latter algorithm enabled him to implement very efficient algorithms in CAYLEY for the induction and restriction of representations from and to subgroups U of G, respectively. They are very useful for computations with matrix generators of a group.

For an idempotent $e = e^2 \neq 0$ of the group algebra FG of a finite group G, the projective FG-module $e\mathrm{F}G$ has endomorphism ring $\mathrm{End}_{\mathrm{F}G}(e\mathrm{F}G) \cong e\mathrm{F}Ge$. Furthermore, for each right FG-module M, there is a right $e\mathrm{F}Ge$-module Me. Of course, it is only necessary to choose idempotents e such that the condensed $e\mathrm{F}Ge$-module Me has dimension $\dim_{\mathrm{F}} Me < \dim_{\mathrm{F}} M$. If by means of the MEAT-AXE or other methods it is possible to find a composition series or the socle of the $e\mathrm{F}Ge$-module Me, then it is sometimes possible to obtain information about the composition series, socle or composition factors of the FG-module M. Recently, Ryba (1990) has successfully applied this method of condensation by constructing a new irreducible 2-modular $G_2(3)$-module of dimension 378. He uses an idempotent e of the form

$$e = \frac{1}{|H|} \sum_{h \in H} h, \quad \text{where } (|H|, \mathrm{char}(\mathrm{F}) = p) = 1.$$

Furthermore, Ryba assumes that the p'-group H acts monomially on the FG-module M, and that $\dim_{\mathrm{F}} M < 1000|H|$. In his case, the module M is an nth wedge power, $\wedge^n W$, of a fairly small FG-module W. According to Ryba, the new condensation program can be applied to a large class of representations M with $\dim_{\mathrm{F}} M \le 10^6$.

Nevertheless, this method is also demanding. Ryba (1990) points out that the user of the condensation program must solve the following preliminary.

PROBLEM 9. Let $g_1, g_2, \ldots, g_r$ be a set of generators of the finite group G represented on the FG-module M by the matrices $\kappa(g_i) \in GL(\dim_{\mathrm{F}} M, \mathrm{F})$, $1 \le i \le r$. Suppose that the generators $h_1, h_2, \ldots, h_t$ of the p'-subgroup H can be written as words in the generators g_i. Then it is necessary to find group elements $v_1, v_2, \ldots, v_s \in G$ written as words in the g_i such that $\{y_1 = ev_1e,\ y_2 = ev_2e, \ldots, y_s = ev_se\}$ generate the F-algebra $e\mathrm{F}Ge$, where

$$e = \frac{1}{|H|} \sum_{h \in H} h.$$

The ($eFGe$)-module structure of Me is then determined by the matrices $\kappa(y_1, \kappa(y_2), \ldots, \kappa(y_s)$.

In the particular case, $G = G_2(3)$, Ryba was able to overcome the difficulty of Problem 9 using a clever trick.

A classical problem of Clifford theory is concerned with the following question: Suppose that U is a normal subgroup of G, and that M is a simple FU-module which is inertial in G. How can the irreducible components of $M^G = M \otimes_{FU} FG$ be efficiently determined? As is well known, this problem is equivalent to that of finding all the irreducible projective representations of G/U (Isaacs, 1976). Recently, Conlon (1990) has given efficient algorithms for the computation of the degrees of the irreducible projective representations of a finite solvable group. These methods may also be applied to the above induction problem provided G/U is a solvable group. Since the construction of the projective representations of a group G depends on the Schur multiplier of G, Holt's algorithm (1984) for calculating the p-primary part of the Schur multiplier of a finite permutation group is relevant here.

5. Groups with a T.I. Sylow p-Subgroup

Blau and the author have studied the modular representations of a finite group G with a T.I. Sylow p-subgroup S, i.e. $S \cap xSx^{-1} = 1$ for all $x \notin N_G(S)$ (Blau & Michler, 1990). For these groups G, the Brauer's height zero conjecture and the Alperin–McKay conjecture in block theory were verified. Unfortunately, the results depend on the classification of the finite groups (Gorenstein, 1982). Furthermore, it was necessary to calculate the character tables of the Sylow normalisers occurring in the possible minimal counter-examples. Most of these tables were calculated by means of the character table algorithms contained in CAYLEY. However, even for these groups G with a T.I. Sylow p-subgroup, the authors were not able to answer Brauer's famous problem which asserts that the number $k(B)$ of irreducible characters of a p-block B of a finite group G with defect group $\delta(B) =_G D$ is bounded by $k(B) \leq |D|$. In fact, the open case where $S = D$, a normal Sylow p-subgroup of G, is extremely hard.

The results of Blau & Michler (1990) suggest the following:

PROBLEM 10. Let G be a finite group with a T.I. Sylow p-subgroup S. Let M be a simple FG-module with vertex $vx(M) =_G S$ and Green correspondent $f(M)$ in $H = N_G(S)$ defined by

$$M\downarrow_H = f(M) \oplus \text{projective},$$

where $f(M)$ is the unique non-projective FH-module of the restriction $M\downarrow_H$. Is the socle $\mathrm{soc}_1[f(M)]$ of $f(M)$ multiplicity free?

In order to show the strength of the algorithms described in the previous sections, the following results of Gollan which also support Problem 10 are now explained (Gollan, 1990).

Let $G = {}^2F_4(2)'$ be the simple Tits group. Its Sylow 5-subgroup S is a T.I. set. By Hiss (1986) the principal 5-block B of G has simple FG-modules of dimensions 1, 26, 27, 78, 109, 351, 460 and 593 over the field $F = GF(25)$ with 25 elements, where 1 denotes the trivial FG-module. Here, the modules of dimensions 26, 27 and 351 are not isomorphic to their respective duals 26*, 27* and 351*. Furthermore, there are non-isomorphic algebraic conjugate modules 109′, 460′ and 593′.

As mentioned in the ATLAS (Conway *et al.*, 1985), the Sylow 5-normaliser $H \cong (\mathbb{Z}_5 \times \mathbb{Z}_5):4A_4$. Hence, H has the following simple FH-modules named according to their dimensions:

$$\mathrm{I}, 1, 1_a, 1_a^*, 1_b, 1_b^*, 2_a, 2_a^*, 2_b, 2_b^*, 2_c, 2_c^*, 3_a, 3_b.$$

By Gollan (1989), the Green correspondents $f(n)$ of the simple FG-modules n have the following socle series:

$$f(26) = 1_a^*, \qquad\qquad f(26^*) = 1_a$$

$$f(27) = \begin{array}{ccccc} & & 1 & & \\ & & 2_a & & \\ & 3_b & & 3_b & \\ 2_a^* & & 2_b & & 2_c^* \\ 1_a & & 1_a^* & & 3_a \\ & 2_b^* & & 2_c & \\ & & 3_b & & \end{array}, \qquad f(27^*) = \begin{array}{ccccc} & & 3_b & & \\ & 2_b & & 2_c^* & \\ 1_a & & 1_a^* & & 3_a \\ & 2_b^* & & 2_c & \\ & & 3_b & & \\ & 2_a & & 2_a^* & \\ & 1 & & 3_b & \end{array}$$

$$f(78) = \begin{array}{ccccc} & & 2_a^* & & \\ & & 3_a & & \\ 2_a & & 2_b^* & & 2_c \\ 1_b & & 1_b^* & & 3_b \\ & 2_b & & 2_c^* & \\ & \mathrm{I} & & 3_a & \\ & 2_a & & 2_a^* & \end{array}$$

$$f(109) = \begin{array}{c} 1_b \\ 2_b \\ 3_a \\ 2_b^* \\ 1_b^* \end{array}, \qquad\qquad f(109') = \begin{array}{c} 1_b^* \\ 2_c^* \\ 3_a \\ 2_c \\ 1_b \end{array}$$

$$f(351) = \begin{array}{ccccccccccccccc} & & & & & & & 3_a & & & & & & & \\ & & & & & 2_a & & 2_b^* & & 2_c & & & & & \\ & & & \mathrm{I} & & 1_b & & 1_b^* & & 3_b & & 3_b & & & \\ 2_a^* & & 2_a^* & & 2_b & & 2_b & & 2_b & & 2_c^* & & 2_c^* & & 2_c^* \\ & 1 & & 1_a & & 1_a^* & & 3_a & & 3_a & & 3_a & & 3_a & \\ & & & \mathrm{I} & & 1_b & & 1_b^* & & 3_b & & 3_b & & & \\ & & & & & 2_a^* & & 2_b & & 2_c^* & & & & & \end{array}$$

$$
f(351^*) = \begin{array}{ccccccccccccc}
 & & & & 2_a & & 2_b^* & & 2_c & & & & \\
 & & \mathrm{I} & & 1_b & & 1_b^* & & 3_b & & 3_b & & \\
 & 2_a^* & & 2_a^* & & 2_b & & 2_b & & 2_c^* & & 2_c^* & \\
1 & & 1_a & & 1_a^* & & 3_a & & 3_a & & 3_a & & 3_a \\
 & 2_a & & 2_a & & 2_b^* & & 2_b^* & & 2_c & & 2_c & \\
 & & \mathrm{I} & & 1_b & & 1_b^* & & 3_b & & 3_b & & \\
 & & & & 2_a^* & & 2_b & & 2_c^* & & & & \\
 & & & & 2_b^* & & 2_c & & 3_a & & & &
\end{array}
$$

$$
f(460) = \begin{array}{cccccc}
 & & 2_a^* & 2_b & & \\
 & 1 & 1_a & 3_a & 3_a & \\
2_a & 2_a & 2_b^* & 2_b^* & 2_c & 2_c \\
\mathrm{I} & 1_b & 1_b^* & 3_b & 3_b & 3_b \\
2_a^* & 2_a^* & 2_b & 2_b & 2_c^* & 2_c^* \\
 & 1 & 1_a^* & 3_a & 3_a & \\
 & & 2_a & 2_b^* & &
\end{array}
$$

$$
f(460') = \begin{array}{cccccc}
 & & 2_a^* & 2_c^* & & \\
 & 1 & 1_a^* & 3_a & 3_a & \\
2_a & 2_a & 2_b^* & 2_b^* & 2_c & 2_c \\
\mathrm{I} & 1_b & 1_b^* & 3_b & 3_b & 3_b \\
2_a^* & 2_a^* & 2_b & 2_b & 2_c^* & 2_c^* \\
 & 1 & 1_a & 3_a & 3_a & \\
 & & & 2_a & 2_c &
\end{array}
$$

$$
f(593) = \begin{array}{ccc}
 & 3_a & \\
2_a & & 2_b^* \\
\mathrm{I} & & 3_b \\
2_a^* & & 2_b \\
 & 3_a &
\end{array}, \qquad
f(593') = \begin{array}{ccc}
 & 3_a & \\
2_a & & 2_c \\
\mathrm{I} & & 3_b \\
2_a^* & & 2_c^* \\
 & 3_a &
\end{array}
$$

Remark 4. Except for the cases (460, 460′), Gollan (1990) has shown that the restriction of the Green correspondent to the Sylow p-subgroup S of G is indecomposable. Thus, the restrictions of the Green correspondents $f(n)$ to S are the sources of the simple FG-modules n. In the cases (460, 460′), this restriction of $f(n)$ decomposes into four summands.

Remark 5. The Sylow 5-subgroup S of the McLaughlin group McL is a T.I. set. With the present methods it is possible to answer Problem 10 for McL. This will be completed in Essen in due course.

If the assertion of Problem 10 is also verified for the McLaughlin group McL, then by Blau & Michler (1990) the only possible counter-examples G to Problem 10 have to involve the simple Janko group J_4 as a composition factor. Since the dimensions of the non-trivial simple FJ_4-modules over the field $F = GF(11)$, all exceed 1332, the only manageable candidate is the simple module of dimension 1333. Of course, the method of section 3 will lead to a new construction of J_4 over $GF(11)$. The restriction of this representation to the Sylow 11-normaliser of J_4 can then be computed by the methods of section 4. At present it cannot be decided whether this example will provide a counter-example or even more evidence for an affirmative answer to the question posed as Problem 10.

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